PHYSICAL REVIEW E **79**, 050106(R) (2009)

Cusps, self-organization, and absorbing states

Juan A. Bonachela,[1] Mikko Alava,[2] and Miguel A. Muñoz[1]
[1]*Departamento de Electromagnetismo y Física de la Materia and Instituto de Física Teórica y Computacional Carlos I, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain*
[2]*Department of Applied Physics, Helsinki University of Technology, HUT-02105, Finland*
(Received 9 October 2008; published 26 May 2009)

Elastic interfaces embedded in (quenched) random media exhibit metastability and stick-slip dynamics. These nontrivial dynamical features have been shown to be associated with cusp singularities of the coarse-grained disorder correlator. Here we show that *annealed* systems with many absorbing states and a conservation law but *no quenched disorder* exhibit identical cusps. On the other hand, similar nonconserved systems in the directed percolation class are also shown to exhibit cusps but of a different type. These results are obtained both by a recent method to explicitly measure disorder correlators and by defining an alternative new protocol inspired by self-organized criticality, which opens the door to easily accessible experimental realizations.

DOI: 10.1103/PhysRevE.79.050106 PACS number(s): 64.60.Ht, 02.50.−r, 05.50.+q, 05.70.Ln

Elastic objects or interfaces evolving in disordered media constitute a simple and unified way to describe the phenomena as different as wetting, dislocations, cracks, domain walls in magnets, or charge-density waves [1]. In all these examples, two conflicting mechanisms compete: the tendency to minimize the elastic energy, leading to flat interfaces, and the propensity to accommodate to local minima of a random pinning potential, giving rise to rough profiles. In this way, many different meta-stable states exist, opening the door to the rich phenomenology characteristic of disordered systems: slow dynamics, creeping, stick-slip motion, avalanches, etc [2]. At *zero temperature* and in the presence of a pulling force, a critical point separates two different phases: (i) a *pinned* one, in which disorder dominates and the interface remains trapped at some local minimum of the random potential, and (ii) a *depinned* phase in which the force overcomes the pinning potential and the interface moves forward. A prominent example characterizing the dynamics of these systems is the quenched Edwards-Wilkinson equation

$$\partial_t h(\mathbf{x},t) = \nu \nabla^2 h(\mathbf{x},t) + F + \eta[\mathbf{x},h(\mathbf{x},t)], \qquad (1)$$

where ν is a constant, $h(\mathbf{x},t)$ is the interface height, F is an external force, and $\eta[\mathbf{x},h(\mathbf{x},t)]$ is a quenched (random-field) noise. The noise correlator is $\langle \eta(\mathbf{x},0)\eta(\mathbf{x}',u)\rangle = \Delta_0(u)\delta(\mathbf{x}-\mathbf{x}')$, where $\Delta_0(u)$ is some fast-decaying function. At a critical force, F_c, Eq. (1) shows a *depinning transition*, representative of a broad universality class. Analytical understanding of Eq. (1) comes from various fronts, including the *functional renormalization group* (FRG) [3,4], in which the disorder correlator, $\Delta_0(u)$, renormalizes as a function. This approach led, some years ago, to a successful computation of critical exponents in an $\epsilon(=4-d)$ expansion up to one-loop order [3]. The emerging FRG fixed-point function for the disorder correlator, $\Delta^*(u)$, presents the peculiarity of having a *cusp singularity at the origin*, i.e., $\Delta^{*'}(0^+) = -\Delta^{*'}(0^-) \neq 0$. Very interestingly, such "cusps," rather than curiosity, have been argued to be essential to capture the physics of disordered systems [2,5–7]. Physically, the underlying idea is that the effective *energy landscape* describing a random media in which an interface advances consists of a series of parabolic wells (representing different metastable states) matching at singular points where the first derivative (i.e., the force) is discontinuous [2,6,7]. In this way, the total random force experienced by the interface as it advances has necessarily a *sawtooth profile*, with linear increases followed by abrupt falls (see Fig. 1). These jumps reflect the change from one metastable state to another; they dominate the statistics of the correlation functions and generate a cusp singularity at the origin of the renormalized disorder correlator. Le Doussal and co-workers [4–6] extended the FRG calculation up to two-loops and developed a strategy to measure disorder correlators, allowing us to test the FRG predictions in computer simulations as follows. In Eq. (1), F is replaced by a confining force $F_{LDW} = m^2[w - h(x,t)]$, derivative of a parabolic potential centered at a $h=w$, where w is a constant. For any w, a stable sample-dependent interface configuration, with average height $\bar{h}$, is found. Then, by slowly increasing w, F_{LDW} grows until, eventually, the interface overcomes a barrier and falls into a new metastable state with larger $\bar{h}$ (giving rise to

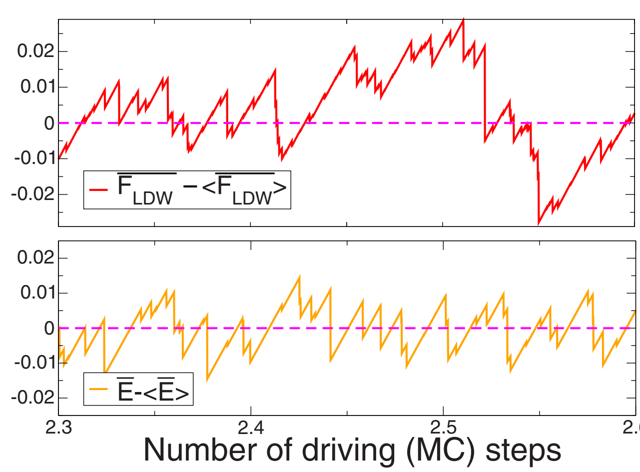

FIG. 1. (Color online) Up (down): Steady-state time series of the spatially averaged force (background field) for a single run of Eq. (1) [Eq. (3)] using the protocol by Le-Doussal-Wiese (using self-organized criticality); the average values have been subtracted in both cases. Linear increases are followed by abrupt falls: the fingerprint of the cusps.

sawtooth profiles). The main breakthrough in [5,6] is to prove that, for a size L, the cumulants of $w-\bar{h}$ can be written as

$$\langle w-\bar{h}(w)\rangle = F_c(m)/m^2,$$

$$\langle\{[w-\bar{h}(w)][w'-\bar{h}(w')]\}\rangle_c = \Delta_m(w-w')/(m^4L^d),$$

$$\langle\{[w-\bar{h}(w)]-[w'-\bar{h}(w')]\}^3\rangle_c = S_m(w-w')/(m^6L^{2d}) \tag{2}$$

and that, taking the limit $m\to 0$, $F_c(0^+)$ converges to the critical force F_c and Δ_{0^+} (respectively, S_{0^+}) coincides with (is a function of) the FRG fixed point, $\Delta^*(u)$ [6,7]. Numerical measurements of the cumulants in Eq. (2) agree nicely with the two-loop FRG predictions [6,7].

In this Rapid Communication, we show that cusps appear also in the apparently very different realm of systems with annealed disorder; systems with many absorbing states (ASs), a conservation law, and no quenched disorder exhibit cusps identical to those of Eq. (1). Instead, similar models with AS in the directed percolation (DP) class, as well as the Bak-Tang-Wiesenfeld sandpile model have also cusps but of different types. Moreover, we introduce an alternative strategy to measure the cusps using self-organized criticality (SOC), i.e., by alternating slow- driving and boundary dissipation. This provides a practical and easy strategy to observe the cusps in numerics and in experiments.

Systems with absorbing states have *annealed* noise and a dynamics which leads to frozen/absorbing microscopic configurations. Well-known examples are DP, the contact process, or the Domany-Kinzel automaton [8]. All these exhibit a transition from an absorbing to an active phase in the very robust DP universality class, represented by the Langevin equation: $\partial_t\rho(\mathbf{x},t)=a\rho-b\rho^2+\nabla^2\rho+\sigma\sqrt{\rho}\eta(\mathbf{x},t)$, where a and b are the parameters, $\rho(\mathbf{x},t)$ is the activity field, and η is a zero-mean Gaussian white noise. The square-root noise ensures the AS condition: fluctuations cease at $\rho=0$. Scaling features different from DP emerge only in the presence of extra symmetries or conservation laws [8,9]. Of particular interest are systems with many AS, in which absorbing configurations have a nontrivial structure represented by a *background field* encoding the likeliness of absorbing configurations to propagate activity when perturbed and allowing for *metastability* to appear. Two main classes of such systems are the following:

(i) The conserved-DP (C-DP) class (which captures the gist of stochastic sandpiles as the *Manna* or the *Oslo* one [10]) is represented by a Langevin equation similar to that of DP but with an extra conservation law [9,11,12]:

$$\partial_t\rho(\mathbf{x},t) = a\rho - b\rho^2 + \gamma\rho E(\mathbf{x},t) + \nabla^2\rho + \sigma\sqrt{\rho}\eta(\mathbf{x},t)$$

$$\partial_t E(\mathbf{x},t) = D\nabla^2\rho, \tag{3}$$

where a, b, D, and γ are parameters and $E(\mathbf{x},t)$ is the (conserved and nondiffusive) background field. Notwithstanding its similarity with the well-behaved DP equation, Eq. (3) has resisted all renormalization attempts and sound analytical predictions are not yet available [13].

(ii) The directed percolation class with many AS [14]. Defined by models as the *pair contact process* [14], this class has no extra symmetry/conservation law with respect to DP. Its corresponding Langevin equation is

$$\partial_t\rho(\mathbf{x},t) = a\rho - b\rho^2 + \gamma\rho\Psi(\mathbf{x},t) + \nabla^2\rho + \sigma\sqrt{\rho}\eta(\mathbf{x},t)$$

$$\partial_t\Psi(\mathbf{x},t) = \alpha\rho - \beta\Psi\rho, \tag{4}$$

where a,b,γ,α and β are parameters. Despite the nontrivial absorbing phase, characterized by the background field $\Psi(\mathbf{x},t)$, Eq. (4) exhibits DP (bulk) criticality [14].

Remarkably, it has been conjectured that Eqs. (1) and (3) are equivalent descriptions of the same underlying physics [11,15]. In spite of the absence of rigorous proof, there is strong theoretical and numerical evidences backing this conjecture [15]. Indeed, there are heuristic arguments which, starting from Eq. (3) and defining a "virtual interface" $H(\mathbf{x},t)=\int_0^t dt'\rho(\mathbf{x},t')$ which encodes the past activity history, lead to Eq. (1) as an effective equation for $H(\mathbf{x},t)$. Note that any past noise trajectory is a frozen variable, allowing us to map present-time annealed noise into quenched disorder. Moreover, the background field $E(\mathbf{x},t)$ in Eq. (3) can be identified with the force $F+\nu\nabla^2 h(\mathbf{x},t)$ in Eq. (1); both encode the propensity to propagate activity or motion at each site [15].

In what follows, we (a) reproduce the results in [6] for Eq. (1) and introduce an alternative protocol to measure them, (b) look for cusps in Eq. (3) and compare them with those in quenched systems to verify if, indeed, they both represent the same physics, and (c) explore whether different types of cusps exist for other AS systems.

(a) We directly integrate a discretized version of Eq. (1) in one dimension, with a force $m^2[w-h(\mathbf{x},t)]$ and periodic boundary conditions [6]. $\eta(\mathbf{x},h)$ takes quenched random values at discrete equispaced values of h and is linearly interpolated in between. To simplify the numerics we use an overall Heaviside step function in the right-hand side (rhs) of Eq. (1) forbidding the interface to move backward [16]. For each w the dynamics eventually reaches a pinned configuration (nonpositive rhs everywhere). By increasing quasistatically the value of w and using Eq. (2), we determine F_c, $\Delta(u)$, and $S(u)$. It is convenient (see [5,6]) to use normalized functions defined by $Y(u/u_\xi)=\Delta(u)/\Delta(0)$ with u_ξ fixed by $\int_0^\infty Y(z)dz=1$ and $Q[\Delta(w)/\Delta(0)]=\int_0^w S(w')dw'/\int_0^\infty S(w')dw'$. The universal functions $Y(z)$ and $Q(y)$ are determined for up to $L=2^{12}$ and $m^2=0.001$. Results are indistinguishable from those in [6] for both functions (see Fig. 2).

We now define an *alternative protocol* to measure the cusps. The basic idea is to let the system self-organize to its critical state by iterating local slow driving and boundary dissipation as done in self-organized criticality [17,18]. Starting from a pinned interface, slow driving is implemented by increasing the force F (which becomes a $\mathbf{x}$-dependent function) at a randomly selected site by a small amount. This is repeated until the interface gets depinned at some point. Then (and only then) the dynamics [Eq. (1)] proceeds. Open boundaries $h(0)=h(L+1)=0$ allow for "dissipation" or force

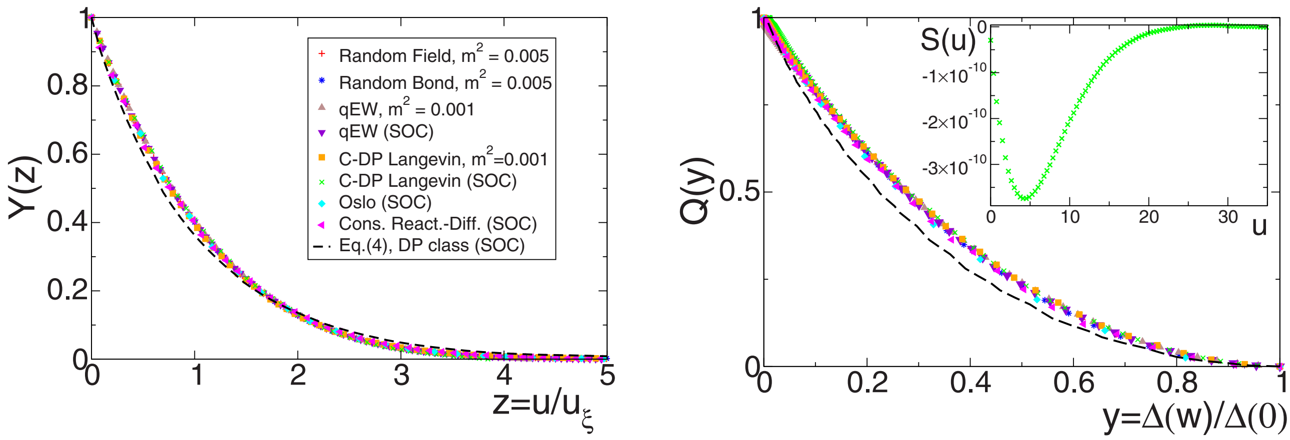

FIG. 2. (Color online) Upper (lower): $Y(z)$ [$Q(y)$] for nine different one-dimensional cases: (i) Eq. (1) for random field and (ii) random bond disorder (from [6]) and (iii) from our own simulations, (iv) self-organized Eq. (1), (v) C-DP [Eq. (3)] (vi) self-organized Eq. (3), (vii) self-organized Oslo model and, finally, (viii) a critical conserved reaction diffusion model. All these curves collapse into a unique one, different from the one obtained for (ix) DP with many AS [Eq. (4)] following the self-organized procedure. Differences are larger for $Q(y)$ than for $Y(z)$. Inset: function $S(u)$, from which $Q(y)$ is determined, for Eq. (3).

reduction (i.e., owing to the open boundaries, the interface develops, on average, a parabolic profile with *negative* average elastic force) inducing the system to get pinned again, and so forth. This driving/dissipation cycle is iterated until a steady state is reached. Then correlations between the spatially averaged (pulling plus elastic) force $\overline{F}+\overline{\nabla^2 h}$, at different driving steps, are measured. Note that this averaged force defines a stochastic (generalized Ornstein-Uhlenbeck) process confined inside a L-dependent potential. As the force balances at each pinned state the corresponding random pinning force, its correlations provide direct access to the quenched-disorder correlations. We have verified that the average pulling force is identical to the critical force F_c [see Eq. (2)] and measured the cumulant ratios; both $Y(z)$ and $Q(y)$ turn out to be indistinguishable from their counterparts above (see Fig. 2). This coincidence follows from the fact that we are just considering two alternative "ensembles" to study Eq. (1) at criticality. In the first, *fixed velocity ensemble*, one considers closed boundaries and lets the velocity go to 0^+ ($m^2 \to 0$). Instead, in the second one, we let the system self-organize by combining slow driving and boundary dissipation at infinitely separated time scales [17,18]. These two paths lead to the same critical properties [19].

(b) We now adapt these two protocols to study systems with many AS. We start analyzing Eq. (3) in one-dimensional lattices (size up to $L=2^{12}$) using the recently proposed efficient integration scheme for Langevin equation. with square-root noise [20]. The system is initialized with arbitrary initial conditions and then it self-organizes to its critical point by means of slow driving and boundary dissipation [17] (second protocol) as follows: when the system is at an AS, the background and the activity field are increased, at a randomly chosen site, by a small amount ($\delta E=\delta\rho=0.1$); then, the dynamics proceeds according to Eq. (3). The open boundaries allow us to decrease the averaged background field and hence the linear term in the activity equation, until eventually a new AS is reached. Then, a new driving step is performed and so on, until a steady state is reached. Figure 1 shows the evolution of $\bar{E}$ (minus its average value) as a function of the number of driving steps in such a state; linear increases are followed by abrupt decays corresponding to dissipative avalanches. From such time series, we measure the three cumulants of the spatially averaged background field (equivalent to the force above) and, from them, the critical point, $Y(z)$ and $Q(y)$. Results for $L=2^{10}$, summarized in Fig. 2, show an excellent agreement with their counterparts obtained for Eq. (1).

To obtain results for Eq. (3) using the strategy in [6], one needs to introduce a slowly moving parabolic potential ($m^2[w-H(\mathbf{x},t)]$ with $H(\mathbf{x},t)=\int_0^t dt' \rho(\mathbf{x},t')$) for the background field and periodic boundaries. This is achieved by including a term $-m^2\rho(\mathbf{x},t)$ in its Langevin Eq. (3) and increasing E by a small constant amount, $m^2\delta w$, after each avalanche. When integrated in time, these two contributions give a slowly moving parabolic potential for E. Additionally, after each avalanche, activity at some random point is created to avoid remaining trapped in the AS. This causes a cascade of rearrangements in the activity and background fields which eventually stops owing to the dissipation in E induced by the negative forcing term ($-m^2\rho$). From the correlations of average background field values in the AS at different steps, we obtain results identical to those reported before for both $Y(z)$ and $Q(y)$ (see Fig. 2).

The (impressive) overlap of whole functions makes a much stronger case for the conjectured equivalence between Eqs. (1) and (3) (and also between the two protocols) than any previous numerical agreement of critical exponents. These conclusions have also been extended to bidimensional systems (not shown).

Let us now study different stochastic sandpile models [10] as well as a *reaction diffusion system* with conservation [9], all of them argued to be in the C-DP class [18] but for which, owing to the discrete nature of particles or grains, the continuous *statistical tilt symmetry* [which leaves the physics of Eq. (1) invariant under the continuous $h(x) \to h(x)+\delta h(x)$ transformation [7]] is broken. Using the self-organized protocol, we produce the cusps reported in Fig. 2. The curves overlap almost perfectly with the previously obtained ones, implying that the continuous statistical tilt symmetry is asymptotically restored and confirming the *universal behavior of stochastic sandpiles and their associated cusps* [21]. Instead, for the *deterministic* Bak-Tang-Wiesenfeld sandpile [22], known to exhibit different critical behavior, we have measured different correlators, with another type of cusps (not shown).

(c) Finally, we apply the previous methods to the DP (with many AS) class [Eq. (4)] for which an effective interfacial description can also be constructed [23]. Results are summarized again in Fig. 2. $Y(z)$ shows a cusp at the origin which differs slightly from the one above; the discrepancy in $Q(y)$ is much more pronounced, confirming that DP exhibits a different type of scaling, and that the cusps are a common trait of systems with many AS. This is a remarkable, so far unveiled, property of DP systems.

In summary, we have shown the presence of identical cusps in systems with quenched disorder, systems with AS and a conservation law, and self-organized stochastic sandpiles. In contrast, different cusps are obtained for systems in the directed percolation class as well as for the deterministic Bak-Tang-Wiesenfeld sandpile. This enriches our present view of universality in stick-slip/avalanching systems and confirms the ubiquitous presence of disorder correlators with cusps in avalanching systems, with or without quenched disorder.

Our protocol to determine the cusps opens a straightforward path to measure them in experiments. Note that for a sandpile it suffices to measure its weight after every avalanche to determine cusp correlators (see [24] for a student laboratory setup). Experimental studies of the mass of granular piles have actually noticed the presence of sawtooth fluctuations (see Fig. 1 of [25]), and for experimental ricepiles, their correlations have been looked-at [26], but without determining the relevant ones. Similarly, cusps could be easily measured in any system, as superconductors [27], in which self-organized criticality has been observed in the laboratory.

Challenging questions remain unanswered: could the FRG cusps be derived from a renormalization group solution of Eq. (3)? Could the DP cusps be determined analytically from Eq. (4)? Are there further systems that exhibit cusps? Answering these questions and studying experimental realizations would greatly enhance our understanding of stick-slip/avalanching dynamics and would provide a rich cross fertilization between interfaces in random media and systems with AS.

[1] D. S. Fisher, Phys. Rep. **301**, 113 (1998).
[2] L. Balents, J. P. Bouchaud, and M. Mézard, J. Phys. I (France) **6**, 1007 (1996).
[3] D. S. Fisher, Phys. Rev. Lett. **56**, 1964 (1986); T. Nattermann, *et al.*, J. Phys. II **2**, 1483 (1992); O. Narayan and D. S. Fisher, Phys. Rev. B **48**, 7030 (1993).
[4] P. Chauve, P. Le Doussal, and K. J. Wiese, Phys. Rev. Lett. **86**, 1785 (2001).
[5] P. Le Doussal, Europhys. Lett. **76**, 457 (2006); A. A. Middleton, P. Le Doussal, and K. J. Wiese, Phys. Rev. Lett. **98**, 155701 (2007).
[6] A. Rosso, P. Le Doussal, and K. J. Wiese, Phys. Rev. B **75**, 220201(R) (2007); P. Le Doussal, and K. J. Wiese, EPL **77**, 66001 (2007).
[7] K. J. Wiese and P. Le Doussal, Markov Processes Relat. Fields **13**, 777 (2007); P. Le Doussal, e-print arXiv:0809.1192.
[8] H. Hinrichsen, Adv. Phys. **49**, 815 (2000).
[9] M. Rossi, R. Pastor-Satorras, and A. Vespignani, Phys. Rev. Lett. **85**, 1803 (2000).
[10] S. S. Manna, J. Phys. A **24**, L363 (1991); K. Christensen, A. Corral, V. Frette, J. Feder, and T. Jossang, Phys. Rev. Lett. **77**, 107 (1996).
[11] A. Vespignani, R. Dickman, M. A. Munoz, and S. Zapperi, Phys. Rev. Lett. **81**, 5676 (1998); Phys. Rev. E **62**, 4564 (2000); R. Dickman *et al.*, Braz. J. Phys. **30**, 27 (2000).
[12] S. Lübeck, Int. J. Mod. Phys. B **18**, 3977 (2004).
[13] There is a (likely to be incomplete) perturbative RG analysis: F. van Wijland, Phys. Rev. Lett. **89**, 190602 (2002).
[14] I. Jensen, Phys. Rev. Lett. **70**, 1465 (1993); M. A. Muñoz, G. Grinstein, R. Dickman, and R. Livi, *ibid.* **76**, 451 (1996); Physica D **103**, 485 (1997); M. A. Muñoz, G. Grinstein, and R. Dickman, J. Stat. Phys. **91**, 541 (1998).
[15] M. Alava and M. A. Muñoz, Phys. Rev. E **65**, 026145 (2002); J. A. Bonachela, H. Chate, I. Dornic, and M. A. Munoz, Phys. Rev. Lett. **98**, 155702 (2007).
[16] H. Leschhorn, Physica A **195**, 324 (1993); H. Leschhorn *et al.*, Ann. Phys. **509**, 1 (1997).
[17] H. J. Jensen, *Self-Organized Criticality* (Cambridge University Press, New York, 1998).
[18] M. Alava, in *Advances in Condensed Matter and Statistical Physics*, edited by E. Korutcheva and R. Cuerno, Nova Science, NY, 2004).
[19] O. Narayan, Phys. Rev. E **62**, R7563 (2000); See also S. Zapperi *et al.*, Phys. Rev. B **58**, 6353 (1998).
[20] I. Dornic, H. Chaté, and M. A. Muñoz, Phys. Rev. Lett. **94**, 100601 (2005); E. Moro, Phys Rev. E **70**, 045102(R) (2004).
[21] Sandpiles can be mapped into interfaces in random media: O. Narayan and A. A. Middleton, Phys. Rev. B **49**, 244 (1994); M. Paczuski, S. Maslov, and P. Bak, Phys. Rev. E. **53**, 414 (1996); M. Alava, J. Phys.: Condens. Matter **14**, 2353 (2002).
[22] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987).
[23] R. Dickman and M. A. Muñoz, Phys. Rev. E **62**, 7632 (2000).
[24] S. K. Grumbacher *et al.*, Am. J. Phys. **61**, 329 (1993).
[25] G. A. Held *et al.*, Phys. Rev. Lett. **65**, 1120 (1990).
[26] A. Malthe-Sorenssen, J. Feder, K. Christensen, V. Frette, and T. Jossang, Phys. Rev. Lett. **83**, 764 (1999).
[27] R. J. Wijngaarden *et al.*, Eur. J. Phys. B **50**, 117 (2006).